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STOCHASTIC PROGRAMMED DESIGN FOR A DETERMINISTIC POSITIONAL DIFFERENTIAL GAME*

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It is shown that under specific sufficiently general conditions the value of a positional differential game can be found from auxiliary programmed constructions which include a suitable random process. The paper is a continuation of the researches in /1—12/.

1. We consider a system described by the differential equation

$$\mathbf{x}^{\cdot} = A(t)\mathbf{x} + B(t)\mathbf{u} + C(t)\mathbf{v}, \quad \mathbf{u} \in P, \quad \mathbf{v} \in Q, \quad t_0 \leqslant t \leqslant \vartheta$$

where $\mathbf{x}$ is the object's n-dimensional phase vector, $\mathbf{u}$ and $\mathbf{v}$ are, respectively, an r-dimensional and an s-dimensional control vectors of the first and second players, $A(t)$, $B(t)$, $C(t)$ are continuous matrix-valued functions, and P and Q are convex compacta. Let the functional

$$\gamma = \gamma(\mathbf{x}(t_*[\cdot]\vartheta), \mathbf{u}(t_*[\cdot]\vartheta), \mathbf{v}(t_*[\cdot]\vartheta)) = \int_{t_*}^{\vartheta} [\omega(t, \mathbf{x}[t]) + \omega_1(t, \mathbf{u}[t]) + \omega_2(t, \mathbf{v}[t])]\, dt + \sigma(\mathbf{x}[\vartheta]) \tag{1.1}$$

be prescribed. Here and below the symbol $\mathbf{y}(t_*[\cdot]\, t^*)$ denotes the function $\{\mathbf{y}[t],\ t_* \leqslant t \leqslant t^*\}$, $[t_*, t^*] \subset [t_0, \vartheta]$; the functions ω, ω_1, ω_2 and σ are continuous; the functions ω and σ satisfy Lipschitz conditions in $\mathbf{x}$. By intent, the first player must minimize functional γ and the second must maximize it. The game is formalized as follows. In $(r+1)$-dimensional and $(s+1)$-dimensional spaces, respectively, we consider the sets

$$P^*(t) = \overline{\mathrm{co}}\, \{\mathbf{u}^* = \{\mathbf{u}, \omega_1(t, \mathbf{u})\},\ \mathbf{u} \in P\}$$
$$Q^*(t) = \overline{\mathrm{co}}\, \{\mathbf{v}^* = \{\mathbf{v}, \omega_2(t, \mathbf{v})\},\ \mathbf{v} \in Q\}$$

and we introduce the new control vectors $\mathbf{u}^* = \{\mathbf{u} = \{u_1^*, \ldots, u_r^*\}, u_{r+1}^*\}$, $\mathbf{v}^* = \{\mathbf{v} = \{v_1^*, \ldots, v_s^*\}, v_{s+1}^*\}$ constrained by the conditions

$$\mathbf{u}^* \in P^*(t), \quad \mathbf{v}^* \in Q^*(t) \tag{1.2}$$

A function which with every possible position $\{t, \mathbf{x}\}$ associates a certain set $S(t, \mathbf{x})$ (possibly, empty) of pairs $\mathbf{s} = \{\mathbf{u}^*, \mathbf{v}^*\}$ of vectors $\mathbf{u}^*$ and $\mathbf{v}^*$ from (1.2), is called a strategy $S(t, \mathbf{x})$. Every absolutely continuous function $\mathbf{x}[t]$, $\mathbf{x}[t_*] = \mathbf{x}_*$ satisfying the condition

$$\mathbf{x}^{\cdot}[t] = A(t)\mathbf{x}[t] + B(t)\mathbf{u}[t] + C(t)\mathbf{v}[t] \tag{1.3}$$

where

$$\{\mathbf{u}^*[t] = \{\mathbf{u}[t], \mathbf{u}_{r+1}^*[t]\},\ \mathbf{v}^*[t] = \{\mathbf{v}[t], \mathbf{v}_{s+1}^*[t]\}\} = \mathbf{s}[t] \in S(t, \mathbf{x}[t]) \tag{1.4}$$

for almost all $t \in [t_*, t^*]$, is called a motion $\mathbf{x}(t_*[\cdot]t^*)$ generated by strategy $S(t, \mathbf{x})$ from the position $\{t_*, \mathbf{x}_*\}$. We assume that

$$\gamma = \gamma(\mathbf{x}(t_*[\cdot]\vartheta), \mathbf{u}^*(t_*[\cdot]\vartheta), \mathbf{v}^*(t_*[\cdot]\vartheta)) = \int_{t_*}^{\vartheta} [\omega(t, \mathbf{x}[t]) + u_{r+1}^*[t] + v_{s+1}^*[t]]\, dt + \sigma(\mathbf{x}[\vartheta]) \tag{1.5}$$

on the motion given. A strategy $S(t, \mathbf{x})$ that satisfies the following condition is called first player's strategy $S_u(t, \mathbf{x})$. For any segment $t_* \leqslant t \leqslant t^*$, position $\{t_*, \mathbf{x}_*\}$ and t-measurable admissible function $\mathbf{v}^*(t_*[\cdot]t^*)$ we can find a t-measurable admissible function $\mathbf{u}^*(t_*[\cdot]\, t^*)$ such that the function $\mathbf{x}(t_*[\cdot]\, t^*)$ satisfying (1.3) and the condition $\mathbf{x}[t_*] = \mathbf{x}_*$ is the motion generated by the strategy $S(t, \mathbf{x}) = S_u(t, \mathbf{x})$, i.e., condition (1.4) with $S = S_u$ is satisfied for it for almost all $t \in [t_*, t^*]$. The second player's strategy $S_v(t, \mathbf{x})$ is defined analogously.

We say that strategies S_u and S_v are compatible if for every choice of $\{t_*, \mathbf{x}_*\}$ and $[t_*, t^*]$ there exists a function $\mathbf{x}(t_*[\cdot]\, t^*)$ which is simultaneously the motion generated by both strategy S_u and strategy S_v. We say that compatible strategies S_u° and S_v° form a saddle point of the game at the minimax of the functional γ of (1.1), (1.5) and form the game's value $\rho^\circ(t, \mathbf{x})$, if for every initial position $\{t_*, x_*\}$ the inequality

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$$\gamma(x(t_*[\cdot]\vartheta),\ u^*(t_*[\cdot]\vartheta),\ v^*(t_*[\cdot]\vartheta)) \leqslant \rho^\circ(t_*, x_*)$$

is valid for every motion $x(t_*[\cdot]\vartheta)$ generated by strategy S_u° and the inequality

$$\gamma(x(t_*[\cdot]\vartheta),\ u^*(t_*[\cdot]\vartheta),\ v^*(t_*[\cdot]\vartheta)) \geqslant \rho^\circ(t_*, x_*)$$

is valid for every motion $x(t_*[\cdot]\vartheta)$ generated by strategy S_v°. So, the equality $\gamma = \rho^\circ(t_*, x_*)$ is fulfilled for the motion generated simultaneously by strategies S_u° and S_v°.

The significance of the formalization given is revealed in terms of the approximate strategies. A function $u(t, x, \varepsilon) \in P$ ($v(t, x, \varepsilon) \in Q$), where $\varepsilon > 0$ is a small parameter, is called the first (second) player's approximate strategy. Suppose that ε, a position $\{t_*, x_*\}$, an interval $[t_*, t^*]$ and a partitioning $\Delta = \{\tau_0 = t_*, \tau_{i+1} > \tau_i, \tau_m = t^*\}$ have been chosen. The absolutely continuous solution of the stepwise equation

$$x_\Delta^{\varepsilon\cdot}[t] = A(t)x_\Delta^\varepsilon[t] + B(t)u(\tau_i, x_\Delta^\varepsilon[\tau_i], \varepsilon) + C(t)v[t]$$
$$x_\Delta^\varepsilon[t_*] = x_*,\quad \tau_i \leqslant t < \tau_{i+1},\quad i = 0, 1, \ldots, m-1$$

where the function $v[t] \in Q$ can be any measurable function, is called the $\{\varepsilon, \Delta\}$-motion $x_\Delta^\varepsilon(t_*[\cdot]t^*)$ generated by strategy $u(t, x, \varepsilon)$. The $\{\varepsilon, \Delta\}$-motion generated by strategy $v(t, x, \varepsilon)$ is defined analogously.

We shall examine only the motions $x(t_*[\cdot]t^*)$ and $x_\Delta^\varepsilon(t_*[\cdot]t^*)$ starting in the regions

$$x[t_*] = x_* \in G[t_*] = \{|x| \leqslant r(t_*)\} \tag{1.6}$$
$$r(t_*) = [r_0 + (f+g)/L]\exp L[t_* - t_0] - (f+g)/L$$
$$f = \max|B(t)u|,\ g = \max|C(t)v|,\ L = \max|A(t)|$$

where $|x|$ is the Euclidean norm of vector x and $|A(t)|$ is the Euclidean norm of matrix $A(t)$. For such motions the inclusion $x[t] \in G[t]$ is valid for all $t \in [t_*, t^*]$. We say that strategy $u(t, x, \varepsilon)$ approximates strategy $S_u(t, x)$ if for any $\zeta > 0$ we can find $\varepsilon(\zeta) > 0$ and $\delta(\zeta, \varepsilon) > 0$ such that for any $\{\varepsilon, \Delta\}$-motion $x_\Delta^\varepsilon(t_*^\varepsilon[\cdot]\vartheta)$ generated by strategy $u(t, x, \varepsilon)$ we can find, when $\varepsilon \leqslant \varepsilon(\zeta)$ and $\max_i(\tau_{i+1} - \tau_i) \leqslant \delta(\zeta, \varepsilon)$, a motion $x(t_*[\cdot]\vartheta)$ generated by strategy $S_u(t, x)$, satisfying the conditions

$$|\gamma(x_\Delta^\varepsilon(t_*^\varepsilon[\cdot]\vartheta),\ u(t_*^\varepsilon[\cdot]\vartheta),\ v(t_*^\varepsilon[\cdot]\vartheta)) - \gamma(x(t_*[\cdot]\vartheta),\ u^*(t_*[\cdot]\vartheta),\ v^*(t_*[\cdot]\vartheta))| \leqslant \zeta$$
$$|t_* - t_*^\varepsilon| \leqslant \zeta,\quad \max_{\tau_* \leqslant t \leqslant \vartheta}|x_\Delta^\varepsilon[t] - x[t]| \leqslant \zeta,\quad \tau_* = \max(t_*^\varepsilon, t_*)$$

The following statement is valid.

Theorem 1.1. The game being examined on the minimax of functional (1.1), (1.5) has the saddle point $\{S_u^\circ, S_v^\circ\}$. The game's value $\rho^\circ(t, x)$ satisfies a Lipschitz condition in t and x in the region $G = \{x \in G[t], t_0 \leqslant t \leqslant \vartheta\}$. The optimal strategies S_u° and S_v° are approximated by suitable optimal strategies $u^\circ(t, x, \varepsilon)$ and $v^\circ(t, x, \varepsilon)$.

The approximate strategies $u^\circ(t, x, \varepsilon)$ and $v^\circ(t, x, \varepsilon)$ are constructed by the scheme in /7,8/ as strategies extremal to the function $\rho(t, w, w_{n+1}) = \rho^\circ(t, w) + w_{n+1}$, where the variables $w[t]$ and $w_{n+1}[t]$ describing the state of the w-model vary in accord with the equations

$$w^\cdot = A(t)w + B(t)u_* + C(t)v_*,\quad u_* \in P,\ v_* \in Q \tag{1.7}$$
$$w_{n+1}^\cdot = \omega(t, w) + \omega_1(t, u_*) + \omega_2(t, v_*) \tag{1.8}$$

Here $\rho(t_*, w_*, w_{n+1*})$ is the exact upper bound of the values of β for which there exists in the w-model (1.7), (1.8) exists a $(\beta - Q_{\{t_*, w_*, w_{n+1*}\}})$-procedure /7/ ensuring the inequality $w_{n+1}[\vartheta] + \sigma(w[\vartheta]) > \beta$ for every motion $\{w[t], w_{n+1}[t]\}$, $t_* \leqslant t \leqslant \vartheta$, generated by this Q-procedure from the initial position $\{t_*, w_*, w_{n+1*}\}$. In this regard the accompanying point /8/ in the w-model $\{w(t, x, \varepsilon), c(t, x, \varepsilon)\}$, corresponding to the current position $\{t, x\}$, is determined, when constructing the strategy $u^c(t, x, \varepsilon)$, from the condition $\min_{\{w, c\}}[\rho^\circ(t, w) - c] = \rho^\circ[t, w(t, x, \varepsilon)] - c(t, x, \varepsilon)$ under the condition

$$|x - w|^2 + c^2 \leqslant \varepsilon(1 + [t - t_0])\exp(3L[t - t_0]) \tag{1.9}$$

or, when constructing the strategy $v^\circ(t, x, \varepsilon)$, from the condition

$$\max_{\{w, c\}}[\rho^\circ(t, w) - c] = \rho^\circ[t, w(t, x, \varepsilon)] - c(t, x, \varepsilon)$$

under condition (1.9). As a result the extremal strategies $u^\circ(t, x, \varepsilon)$ and $v^\circ(t, x, \varepsilon)$ are determined from the conditions

$$\langle \mathbf{u}^\circ(t,\mathbf{x},\varepsilon)\cdot[\mathbf{x}-\mathbf{w}(t,\mathbf{x},\varepsilon)]\rangle + c(t,\mathbf{x},\varepsilon)\,\omega_1(t,\mathbf{u}^\circ(t,\mathbf{x},\varepsilon)) = \min_{\mathbf{u}\in P}\,\mathrm{Idem}\,(\mathbf{u}^\circ(t,\mathbf{x},\varepsilon)\to\mathbf{u})$$

$$\langle \mathbf{v}^\circ(t,\mathbf{x},\varepsilon)\cdot[\mathbf{x}-\mathbf{w}(t,\mathbf{x},\varepsilon)]\rangle + c(t,\mathbf{x},\varepsilon)\,\omega_2(t,\mathbf{v}^\circ(t,\mathbf{x},\varepsilon)) = \min_{\mathbf{v}\in Q}\,\mathrm{Idem}\,(\mathbf{v}^\circ(t,\mathbf{x},\varepsilon)\to\mathbf{v})$$

where $\langle\mathbf{a}\cdot\mathbf{b}\rangle$ is the scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$. Here and further the Idem in an equality's right-hand side denotes an expression coinciding with this equality's left-hand side with the change of symbols indicated within the parentheses.

The strategies $S_u^\circ(t,\mathbf{x})$ and $S_v^\circ(t,\mathbf{x})$ are determined as follows. For a current position $\{t,\mathbf{x}\}$, the strategy $S_u^\circ(t,\mathbf{x})$ fixes the set of all pairs $s=\{\mathbf{u}^*,\mathbf{v}^*\}$, $\mathbf{u}^*\in P^*(t)$, $\mathbf{v}^*\in Q^*(t)$, satisfying the condition

$$\overline{\lim_{\tau\to t-0}}\,\frac{\rho^\circ(\tau,\mathbf{y}[\tau])-\rho^\circ(t,\mathbf{x})}{\tau-t}+\omega(t,\mathbf{x})+u_{r+1}^*+v_{s+1}^*\leqslant 0 \qquad (1.10)$$

while strategy $S_v^\circ(t,\mathbf{x})$ fixes for a position $\{t,\mathbf{x}\}$ the set of all pairs $s=\{\mathbf{u}^*,\mathbf{v}^*\}$, $\mathbf{u}^*\in P^*(t)$, $\mathbf{v}^*\in Q^*(t)$, satisfying the condition

$$\underline{\lim_{\tau\to t-0}}\,\frac{\rho^\circ(\tau,\mathbf{y}[\tau])-\rho^\circ(t,\mathbf{x})}{\tau-t}+\omega(t,\mathbf{x})+u_{r+1}^*+v_{s+1}^*\geqslant 0 \qquad (1.11)$$

Here $\mathbf{y}[\tau]$ is a function defined by the equality

$$\mathbf{y}[\tau]=\mathbf{x}+(\tau-t)\,[A(t)\,\mathbf{x}+B(t)\,\mathbf{u}+C(t)\,\mathbf{v}],\ \tau\leqslant t$$

Here, obviously, conditions (1.10) and (1.11) replace the well-known dynamic programming relations /13,14/ which would holds in the case of a differentiable value $\rho^\circ(t,\mathbf{x})$ satisfying the partial differential equation of the dynamic programming method.

2. The construction of the game's value $\rho^\circ(t,\mathbf{x})$ by the Q-procedure indicated in Sect.1 is not effective in general. Therefore, neither is the construction of strategies $\mathbf{u}^\circ(t,\mathbf{x},\varepsilon)$, $\mathbf{v}^\circ(t,\mathbf{x},\varepsilon)$, $S_u^\circ(t,\mathbf{x})$, $S_v^\circ(t,\mathbf{x})$ by this means. The method of constructing the game's value $\rho^\circ(t,\mathbf{x})$ and the optimal strategies on the basis of auxiliary programmed constructions /6,8/ is more effective. However, this method yields the required solution only under definite regularity conditions /8/. Below we describe a certain development of the method of programmed constructions, which permits us to cover a wider circle of problems. However, a certain additional element, in the form of a suitable probability process, is introduced into the auxiliary programmed constructions being developed. We shall assume that the functions $\omega(t,\mathbf{x})$ and $\sigma(\mathbf{x})$ are convex in $\mathbf{x}$.

Thus, we consider a $\mathbf{w}^*$-model whose current state $\mathbf{w}^*=\{\mathbf{w}=\{w_1^*,\ldots,w_n^*\},\ w_{n+1}^*\}$ is described, in accord with (1.7), (1.8), by the equations

$$\mathbf{w}^\cdot=A(t)\,\mathbf{w}+B(t)\,\mathbf{u}+C(t)\,\mathbf{v},\quad w_{n+1}^{*\cdot}=\omega(t,\mathbf{w})+u_{r+1}^*+v_{s+1}^*,\quad \mathbf{u}^*\in P^*(t),\quad \mathbf{v}^*\in Q^*(t)$$

Suppose that some initial position $\{t_*,\mathbf{w}_*^*\}=\{t_*,\{\mathbf{w}_*,0\}\}$, $t_*\leqslant\vartheta$, $\mathbf{w}_*\in G[t_*]$ has been chosen. We partition the interval $[t_*,\vartheta]$ by the points $t_i=t_*+[\vartheta-t_*]\cdot(i-1)/k$, $i=1,2,\ldots,k$, where k is some sufficiently large integer. We consider a sequence ξ of independent vector-valued random variables $\{\xi_j^{(i)},\ j=1,2,\ldots,n\}$, $i=1,2,\ldots,k$. Each of the variables $\xi_j^{(i)}$ can take one of the two values $\xi_j^{(i)+}=1$ and $\xi_j^{(i)-}=-1$ with equal probabilities $p^+=1/2$ and $p^-=1/2$. A function of t and $\xi=\{\xi_j^{(i)}\}$ with values $\mathbf{v}^*(t_i,\xi)\in Q^*(t_i)$ is called a stochastic nonanticipatory program $\mathbf{v}^*(t,\xi)$; it possesses the property that for $t_i\leqslant t<t_{i+1}$, $i=1,2,\ldots,k$, $t_{k+1}=\vartheta$ we have $\mathbf{v}^*(t,\xi)=\mathbf{v}^*(t_i,\xi[t_*,t_i])$, where the symbol $\xi[t_*,t_i]$ denotes the realization $\{\xi_j^{(s)},\ j=1,2,\ldots,n,\ s=1,2,\ldots,i\}$. The stochastic nonanticipatory program $\mathbf{u}^*(t,\xi)\in P^*(t)$ is defined analogously.

Suppose that an initial position $\{t_*,\mathbf{w}_*\}$ has been given and that a specific value of k and a pair of programs $\{\mathbf{u}^*(\cdot,\cdot),\mathbf{v}^*(\cdot,\cdot)\}$ have been chosen. These data define a random process $\mathbf{w}(t_*[\cdot,\xi]\,\vartheta)$ which is a stepwise solution of the differential equation

$$\mathbf{w}^\cdot=A(t)\,\mathbf{w}+B(t)\,\mathbf{u}(t,\xi)+C(t)\,\mathbf{v}(t,\xi) \qquad (2.1)$$

with the initial condition $\mathbf{w}[t_*]=\mathbf{w}_*$. This process $\mathbf{w}(t_*[\cdot,\xi]\,\vartheta)$ and the controls $\mathbf{u}_{r+1}^*(t,\xi)$, $\mathbf{v}_{s+1}^*(t,\xi)$ determine the random value of the functional $\gamma(\xi)$ in (1.5):

$$\gamma(\xi)=\gamma(\mathbf{w}(t_*[\cdot,\xi]\,\vartheta),\ \mathbf{u}^*(t_*[\cdot,\xi]\,\vartheta),\ \mathbf{v}^*(t_*[\cdot,\xi]\,\vartheta))=\mathbf{w}_{n+1}^*[\vartheta]+\sigma(\mathbf{w}[\vartheta]) \qquad (2.2)$$

We consider the function

$$\rho_*(t_*,\mathbf{w}_*)=\lim_{k\to\infty}\,\max_{\mathbf{v}^*(\cdot,\cdot)}\,\min_{\mathbf{u}^*(\cdot,\cdot)}\,M\{\gamma(\xi)\} \qquad (2.3)$$

where the symbol $M\{\gamma\}$ denotes the mathematical expectation. The definition (2.3) of the function $\rho_*(t, \mathbf{w})$ is well posed. As a matter of fact, the minimum and the maximum in the right-hand side of (2.3) are actually reached on certain programs $\mathbf{u}^*(\cdot,\cdot)$ and $\mathbf{v}^*(\cdot,\cdot)$ since $M\{\gamma(\xi)\}$ is a continuous function of a finite number of variables, specified on a compactum. The existence of the limit in (2.3) is established during the proof of the next Theorem 2.1.

Theorem 2.1. The function $\rho_*(t, \mathbf{w})$ in (2.3) is the value $\rho^\circ(t, \mathbf{w})$ of the positional differential game considered in Sect.1.

The theorem is proved as follows. In the region

$$|\mathbf{z}| \leqslant 2r(\vartheta),\ t_* \leqslant t \leqslant \vartheta \tag{2.4}$$

where $r(\vartheta)$ is computed by (1.6), we construct the function

$$H_\alpha(\mathbf{p}, \mathbf{z}, t) = \min_{\mathbf{u}^* \in P^*(t)} \max_{\mathbf{v}^* \in Q^*(t)} [\langle \mathbf{p} \cdot [A(t)\mathbf{z} + B(t)\mathbf{u} + C(t)\mathbf{v}]\rangle + \omega(t, \mathbf{z}) + u^*_{r+1} + v^*_{s+1} - \alpha|\mathbf{v}^*|^2]$$

where α is some small positive number. Further, we construct the function $F_\alpha(\mathbf{p}, \mathbf{z}, t)$ which has derivatives of all orders, satisfies a Lipschitz condition in the first argument and vanishes outside a sufficiently large region G^* in space $\{t, \mathbf{z}\}$, containing region (2.4). In addition, let the condition

$$|H_\alpha(\mathbf{p}, \mathbf{z}, t) - F_\alpha(\mathbf{p}, \mathbf{z}, t)| \leqslant \alpha$$

be fulfilled for all values of arguments $\mathbf{p}$ and $\mathbf{z}, t$ from region (2.4). Let us consider the partial differential equation for a certain function $\rho_\alpha(t, \mathbf{z})$:

$$\frac{\partial\rho_\alpha}{\partial t} + \frac{\alpha^2}{2}\sum_{i=1}^{n} \frac{\partial^2\rho_\alpha}{\partial z_i^2} + F_\alpha(\mathrm{grad}_z\,\rho_\alpha, \mathbf{z}, t) = 0 \tag{2.5}$$

Let $\sigma(\mathbf{z}, \alpha)$ be a function convex in $\mathbf{z}$ for $|\mathbf{z}| \leqslant 2r(\vartheta)$, having derivatives of all orders, satisfying the condition $|\sigma(\mathbf{z}) - \sigma(\mathbf{z}, \alpha)| \leqslant \alpha$ when $|z| \leqslant 2r(\vartheta)$ and vanishing for all sufficiently large values of $|\mathbf{z}|$. Under the boundary condition

$$\rho_\alpha(\vartheta, \mathbf{z}) = \sigma(\mathbf{z}, \alpha) \tag{2.6}$$

Eq.(2.5) has /15/ a solution $\rho_\alpha(t, \mathbf{z})$ which in any preselected region $|\mathbf{z}| \leqslant R$, $t_* \leqslant t \leqslant \vartheta$ has the continuous partial derivatives $\partial\rho_\alpha/\partial t$, $\partial\rho_\alpha/\partial z_i$, $\partial^2\rho_\alpha/\partial z_i\partial z_j$, $i, j = 1, \ldots, n$. Similary as in /16/, we can verify that the limit relation

$$\lim_{\alpha\to 0} \rho_\alpha(t_*, \mathbf{w}_*) = \rho^0(t_*, \mathbf{w}_*) \tag{2.7}$$

is valid for any position $\{t_*, \mathbf{w}_*\}$ from the region $|\mathbf{w}_*| \leqslant r(t_*)$, $t_0 \leqslant t_* \leqslant \vartheta$

We choose some subsequence of numbers $\{k_j, j = 1, 2, \ldots\}$ for which the limit

$$\lim_{k_j\to\infty} \max_{\mathbf{v}^*(\cdot,\cdot)} \min_{\mathbf{u}^*(\cdot,\cdot)} M\{\gamma(\xi)\} = \rho^*(t_*, \mathbf{w}_*) \tag{2.8}$$

exists. We prescribe a certain value $\varepsilon > 0$. For some value k_j we choose some pair of programs $\{\mathbf{v}^*(\cdot,\cdot), \mathbf{u}^*(\cdot,\cdot)\}$, satisfying the condition

$$M\{\gamma(\xi)\} \leqslant \rho^*(t_*, \mathbf{w}_*) + \varepsilon \tag{2.9}$$

where the random variable $\gamma(\xi)$ of (2.2) is determined by the random solution $\mathbf{w}(t_*[\cdot, \xi]\vartheta)$ of Eq.(2.1) and by the controls $u^*_{r+1}(t, \xi)$, $v^*_{s+1}(t, \xi)$. For any $\varepsilon > 0$ we can find $k(\varepsilon)$ such that when $k_j \geqslant k(\varepsilon)$ we can find, for every program $\mathbf{v}^*(\cdot,\cdot)$, a program $\mathbf{u}^*(\cdot,\cdot)$ such that condition (2.9) is fulfilled; this follows from (2.8). We associate the program pair $\{\mathbf{v}^*(\cdot,\cdot), \mathbf{u}^*(\cdot,\cdot)\}$ chosen with the random motion $\mathbf{z}(t_*[\cdot, \xi, \alpha]\vartheta)$, $\mathbf{z}[t_*, \xi, \alpha] = \mathbf{w}_*$, generated by it, being the stepwise solution of the stochastic differential equation ($\delta(t)$ is the Dirac δ-function)

$$\dot{\mathbf{z}} = A(t)\mathbf{z} + B(t)\mathbf{u}(t, \xi) + C(t)\mathbf{v}(t, \xi) + \sum_{t_* \leqslant t_i \leqslant t} \alpha[(\vartheta - t_*)/k_j]^{1/2}\xi^{(i)}\delta(t - t_i) \tag{2.10}$$

$$(k_j \geqslant k(\varepsilon),\ \xi^{(i)} = \{\xi_j^{(i)}, j = 1, 2, \ldots, n\})$$

This motion $z(t_*[\cdot, \xi, \alpha]\vartheta)$ generates a certain stochastic nonanticipatory program $\mathbf{v}^*(\cdot,\cdot,\alpha)^*$ determined from the condition

$$\mathbf{v}^*(t, \xi, \alpha)^* = \mathbf{v}^*(t_i, \xi, \alpha)^*,\ t_i \leqslant t < t_{i+1} \tag{2.11}$$

$$\langle \mathrm{grad}_z\,\rho_\alpha(t_i, z[t_i, \xi, \alpha]) \cdot C(t_i)\mathbf{v}^*(t_i, \xi, \alpha)^*\rangle + v^*_{s+1}(t_i, \xi, \alpha)^* - \alpha|\mathbf{v}^*(t_i, \xi, \alpha)^*|^2 = \max_{\mathbf{v}^* \in Q^*(t_i)} \mathrm{Idem}\,(\mathbf{v}^*(t_i, \xi, \alpha)^* \to \mathbf{v}^*)$$

Under the conditions introduced such a program $v^*(\cdot,\cdot,\alpha)^*$ from (2.11) is unique. In its own turn we associate with this program a program $u^*(\cdot,\cdot,\alpha)^*$ such that condition (2.9) is fulfilled for the pair $\{u^*(\cdot,\cdot,\alpha)^*, v^*(\cdot,\cdot,\alpha)^*\}$. In such a way, by analogy with the procedure from /6,17,18/, we obtain a many-valued mapping of all program pairs $\{u^*(\cdot,\cdot,\alpha), v^*(\cdot,\cdot,\alpha)\}$ satisfying condition (2.9) onto a subset $\{\{u^*(\cdot,\cdot,\alpha)^*, v^*(\cdot,\cdot,\alpha)^*\}\}$ of the same program pairs. As in /17,18/, we can verify that this mapping has a fixed point. Let it be the program pair $\{u^*(\cdot,\cdot,\alpha)_*, v^*(\cdot,\cdot,\alpha)_*\}$. We consider the motion $z(t_*[\cdot,\xi,\alpha]\vartheta)_*$ generated by this program pair as a solution of the stochastic differential Eq.(2.10) with $z[t_*,\xi,\alpha]=w_*$. For this motion the controls $v^*(t_i,\xi,\alpha)_*$ are determined from (2.11). But then, relying on the fact that the function $\rho_\alpha(t,z)$ is a solution of differential Eq.(2.5) with the boundary condition (2.6), by arguments customary to the dynamic programming method, we obtain the estimate

$$M\{\gamma(\xi,\alpha)_*\} \geqslant \rho_\alpha(t_*,w_*) - \eta(\alpha,k_j) \tag{2.12}$$

where

$$\gamma(\xi,\alpha)_* = \gamma(z(t_*[\cdot,\xi,\alpha]\vartheta)_*, u^*(\cdot,\cdot,\alpha)_*, v^*(\cdot,\cdot,\alpha)_*) \text{ and } \eta(\alpha,k_j)\to 0 \text{ as } k_j\to\infty, \alpha\to 0.$$

On the other hand, let us consider the motion $w(t_*[\cdot,\xi,\alpha]\vartheta)_*$ generated by the same program pair $\{u^*(\cdot,\cdot,\alpha)_*, v^*(\cdot,\cdot,\alpha)_*\}$, but now as a solution of the stochastic differential Eq. (2.1). Condition (2.9) with $\gamma(\xi)=\gamma(\xi,\alpha)=\gamma(w(t_*[\cdot,\xi,\alpha]\vartheta)_*, u^*(\cdot,\cdot,\alpha)_*, v^*(\cdot,\cdot,\alpha)_*)$ is valid for this motion. As the same time, the relation

$$|M\{\gamma(\xi,\alpha)\} - M\{\gamma(\xi,\alpha)_*\}| \leqslant \zeta(\alpha,k_j) \tag{2.13}$$

where $\zeta(\alpha,k_j)\to 0$ as $k_j\to\infty$, $\alpha\to 0$, is valid for the quantities $M\{\gamma(\xi,\alpha)\}$ in (2.9) and $M\{\gamma(\xi,\alpha)_*\}$ in (2.12) obtained thus. Now allowing for (2.7), (2.9), (2.12), (2.13), we obtain the inequality

$$\rho^\circ(t_*,w_*) \leqslant \rho^*(t_*,w_*) \tag{2.14}$$

We establish the opposite inequality

$$\rho^\circ(t_*,w_*) \geqslant \rho^*(t_*,w_*) \tag{2.15}$$

if for the given program $v^*(t,\xi)$ we construct a stochastic nonanticipatory program $u^*(t,\xi)$ over the steps $t_i\leqslant t<t_{i+1}$, having chosen the controls $u[t_i,\xi]=u^\circ[t_i,w[t_i,\xi[t_*,t_{i-1}]],\varepsilon]$ in accordance with the optimal approximate strategy $u^\circ(t,w,\varepsilon)$. Inequalities (2.14) and (2.15) can be obtained for any analogous sequence $\{k_j\}$, for which limit (2.8) exists. From (2.14) and (2.15) it follows that every such limit $\rho^*(t_*,w_*)$ must coincide with the game's value $\rho^\circ(t_*,w_*)$. Hence it follows that limit (2.3) indeed exists and that this limit $\rho_*(t_*,w_*)$ actually equals the game's value $\rho^\circ(t_*,w_*)$. This completes the proof of Theorem 2.1.

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